Implications of Symmetry in Inline and Karman Shedding Mode Interaction

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対称性におけるインラインとカルマン渦放出モードの相互作用に関する研究
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概要:
本論文は強制加振下での円柱後流についての群理論を用いた解析と数値の実験結果を示す。対象として外部からの撮動を受けるカルマン渦（K mode）の応答について調べ、ここでは鏡映対称撮動（S mode）と非対称撮動（K1 mode）について考えている。撮動は供試円柱を機械的に振動させることにより発生させた。

実験は小さな風洞で行った。カルマン渦放出周波数に対する加振の比によって、瞬時に空間を固定した形で、モードロック状態を観測することが出来た。カルマン渦周波数での非対称調和強制振動（モード K1＝K）はカルマンモードを強めた。高調和強制力（モード K1 － K）の場合はカルマンモード K にはほとんど影響がでなかったが、離調効果を確認した。他方で、K1 モードの分解調波（S mode, K mode）での強制加振は K モードに対し K1 モード調波の強い応答を伴い大きく作用した。S モードでの分解調波励振は K モードに対し減衰効果となった。また他方で調和と高調和強制力は元の K モードを破壊し二倍周期での不安定性を引き起こすことが分かった。

群理論によるアプローチを用いると、S/K1 モードと K モードの相互作用を支配する一般的な振動方程式が誘導することが出来る。その形式の性質について解析することを数値の実験結果を解釈する上で手助けとなる。まず、K1/K モードの相互作用において、共通なサブグループに対する対称性のある対称性によって、実験で観測された相関と相互の周波数比での大きな共振について説明することができる。周波数比においては K1 と K モードは対称性において互換性がないことが示され、二つのモードは共通の対称なサブグループを持たない。それゆえ、一般的の安定定常モードよりも、むしろ全体の対称性の破綻によって引き起こされる不確定のそれがより起こりやすいと期待される。S/K モードに対しては、最初の結果を繰り返す。すなわち、理論上カルマンモードは二倍周期の不安定性により破壊されることが示されている。この効果は S モード周波数に対してはカルマンモード周波数の二倍高い場合に引き起こされる。

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1. INTRODUCTION

The dynamics of a cylinder wake subjected to forced excitation are investigated in this work. Forced excitation is a typical approach taken to help probe into the wake dynamics by introducing known perturbations and observing the result. Classical works by Stansby (1976), Naudascher (1986), Williamson & Roshko (1988), Ongoren & Rockwell (1988), Williams et al. (1992) and Krishnamoorthy et al. (2001) typify this approach. Lateral (cross-flow) and inflow forced oscillation of a cylinder was performed over a range of frequencies/wavelengths and amplitudes. In this parameter space, complex shedding modes synchronized to the cylinder oscillations were discovered.

Of particular interest to the present work are the findings of Williamson and Roshko (1988), Ongoren and Rockwell (1988) and Williams et al. (1992), who make particular reference to symmetry aspects of the ‘response’ of the forced wake. Williamson and Roshko identified particular wake modes well defined by symmetries for transverse tube excitation; the modes were labeled 2S, 2P, P+S and 2P+2S based on the number and combination of vortices shed per forcing cycle. Similar modes were identified by Ongoren and Rockwell who used the notation A-I → A-IV for the four ‘antisymmetrical’ modes found for lateral cylinder excitation. In Williams et al.’s experiments, inline forcing was found to significantly affect the mean velocity profile in the wake. Modes corresponding to frequency sum and differences were observed. Period-doubling was also found to occur during symmetric forcing at intermediate forcing amplitudes. The authors came up with ‘simple’ symmetry rules to describe the fundamental wake structure differences for symmetric and antisymmetric forcing.

Symmetrical excitation has been investigated both experimentally and theoretically by Mureithi et al. (2002). Period doubling as observed by Williams et al. was confirmed. Furthermore, a theoretical analysis showed that period-doubling is the dominant phenomenon for inline excitation. The main results are highlighted in this paper.

We present here further experimental results on asymmetrical (K1 mode) excitation of the Karman (K) mode in addition to results on the S mode excitation. Subharmonic and superharmonic excitations are considered. Using symmetry concepts, the general form of the amplitude equations governing the coupling between the Karman, K, mode and forcing modes K1, and S are derived. Based on the resulting equations, certain qualitative conclusions can be drawn regarding the expected interaction between induced flow perturbations and the existing Karman mode. The conclusions are then tested against experimental findings.

Much work on vortex shedding control has been done by numerous researchers (e.g. Tokumaru & Dimotakis (1991), Ffowcs & Zhao (1989), Roussopoulos (1993), Park et al. (1994), and many others). Both experimental and computational approaches have been used. In the present paper, we consider the control problem, directly at the level of the amplitude equations. The goal is to draw some general conclusions regarding the controllability of the initiation of vortex shedding (bifurcation control) as well as control of the shedding process itself. Control inputs considered are the perturbations associated with the S and K1.
forcing modes.

2. WAKE SYMMETRY CONSIDERATIONS AND AMPLITUDE EQUATIONS

In this section the basic concepts used to represent the flow field are presented. This is then followed by a derivation of the mode interaction amplitude equations. Rather than taking an analytical approach, thus starting with the Navier Stokes equations, a geometric approach is taken. To this end, it is necessary to idealize the wake structure as follows. The flow symmetries considered apply in a region slightly downstream of the cylinder (see box in Fig.1). Close to zero flow velocity (laminar flow), prior to the onset of Karman shedding, the cylinder ‘wake’ has reflection symmetry, $Z_2(\kappa)$, relative to the flow direction $x$, Fig.1 (top figure). The wake is also (locally) $x$-translation invariant, giving $SO(2)$ symmetry. Next, let the local flow velocities in the inflow, $x$- and transverse, $y$-, directions be $u(x,y,t)$ and $v(x,y,t)$ respectively. The 2D flow symmetries above mean the following relations hold:

$$
\begin{align*}
  u(x,y,t) &= u(x+l,y,t) \\
  u(x,y,t) &= u(x,-y,t) \\
  v(x,y,t) &= 0 .
\end{align*}
$$

Based on the foregoing, the base symmetry considered is $\Gamma = Z_2(\kappa) \times SO(2) = O(2)$. From a geometrical view point, dynamic instability of the base flow corresponds to symmetry breaking. The post instability flow structure will have a reduced symmetry, which is always a subgroup of the base symmetry $\Gamma$; two examples are shown in Fig.1. Instabilities which lead to periodicity in the flow (Hopf bifurcations) introduce a new temporal symmetry, $S^1$, reflecting the time periodicity. The spatial and temporal symmetries may ‘interact’ to give more complex spatial-temporal symmetries. As shown below, the Karman mode symmetry is an example of a spatio-temporal symmetry.

2.1 Wake Mode Symmetries

The flow field behind the cylinder may be expressed (approximately) in terms of a finite number of dominant wake modes (patterns). The Karman shedding mode is designated $K$. A second well known mode is the reflection symmetric shedding mode $S$. Assuming that all other modes (besides modes $S,K$) are stable, the $x$-direction velocity perturbations, for instance, may be expressed as

$$
\begin{align*}
  u(x,y,t) &= S(t)\psi_S(y)e^{i(\lambda_Sx+\omega_S t)} \\
  &\quad + K(t)\psi_K(y)e^{i(\lambda_Kx+\omega_K t)} + \text{complex conjugate}
\end{align*}
$$

where $S(t)$ and $K(t)$ are the complex mode amplitudes for mode $S$ and $K$, respectively, and $\lambda_S,\lambda_K$ the corresponding wave numbers. For the Karman mode $K$, for instance, the flow field is

$$
\begin{align*}
  u(x,y,t) &= u(x,-y,t+\tau_K/2) = u(x+\lambda_K/2,-y,t) \\
  v(x,y,t) &= -v(x,-y,t+\tau_K/2) = -v(x+\lambda_K/2,-y,t) .
\end{align*}
$$

(3)

This modal symmetry may be compactly expressed as

$$
\Gamma_K = Z_2(\kappa,\pi) \times S^1 \quad (4a)
$$

$\Gamma_S$ and $\Gamma_K$ must be subgroups of the base symmetry.
The function $\gamma$ symmetry is now given by the pair $(\gamma, \pi)$ where

$$\Gamma = \left\{ \gamma \mid \gamma g(x) = g(\gamma x) \right\}. \tag{7}$$

It is evident that given a function $g(x)$ determining the group $\Gamma$ is feasible. It also turns out that given a symmetry group $\Gamma$, it can be shown that there exists a finite set of basis functions ($\Gamma$-equivariant polynomials) that generates the complete set of $\Gamma$-equivariant functions; proof of this is given by the Hilbert-Weyl (HW) theorem (Theorem 4.2 in Golubitsky et al., 1988). The HW theorem provides us with a powerful tool for the derivation of amplitude equations (mode interaction equations) when the underlying symmetry is known.

The base symmetry group we consider here is $\Gamma = Z_2(\kappa) \times SO(2)$. The symmetry group $Z_2(\kappa) = \{ I, \kappa \} ; I$ is the identity element while the element $\kappa$ acts as a reflection. Equivariance relative to $Z_2(\kappa)$ implies that the system of equations should be invariant to reflection about the $x$-axis or the inversion $y \to -y$. An element $\theta$ of the symmetry $SO(2)$ on the other hand acts as a translation $x \to x + \frac{\lambda \theta}{2\pi}$. These symmetry actions have important consequences for the functional form of the terms in equation (2). Since the base flow can be expressed as an infinite series expansion of terms such as those in equation (2), it is clear that each one of these terms must satisfy the base symmetry $\Gamma$. As an example, consider the first term in (2), $u_1(x,y,t) = S(t)\Psi_3(y)e^{i(\lambda x + \omega t)}$. Reflection invariance of this term implies that

$$u_1(x,y,t) = \kappa \circ [u_1(x,y,t)] = u_1(x,-y,t). \tag{8}$$

Action on the modal representation is

$$\kappa \circ [S(t)\Psi_3(y)e^{i(\lambda x + \omega t)}] = (\kappa \circ [S(t)])(\kappa \circ [\Psi_3(y)])(e^{i(\lambda x + \omega t)}). \tag{9}$$

Due to reflection symmetry $\kappa \circ [\Psi_3(y)] = \Psi_3(-y) = \Psi_3(y)$. We conclude, therefore, that reflection acts as the identity on the mode $S$ amplitude, $S(t)$.

Following the same procedure for the second term.
in equation (2), we can show that the reflection action on the amplitude pair is
\[ \kappa \circ [(S, K)] = (S, -K). \]  
\hspace{1cm} (10a)

Similarly, an element \( \theta \) of SO(2) can be shown to act as
\[ \theta \circ [(S, K)] = (e^{im\theta}S, e^{in\theta}K), \quad \frac{m}{n} = \frac{\lambda_s}{\lambda_r}. \]  
\hspace{1cm} (10b)

The symmetry actions (10(a,b)) significantly limit the form of the interaction functions. Suppose the interaction equations are expressed as the Poincare maps (discrete equations)
\[ S_{n+1} = \Phi_S(S_n, K_n) \]
\[ K_{n+1} = \Phi_K(K_n, S_n). \]  
\hspace{1cm} (11)

This helps us eliminate the extra time variation introduced by the Hopf bifurcation. Next, consider the action of the reflection symmetry on the new amplitudes \( S_{n+1}, K_{n+1} \). Since \( \kappa \circ (S_{n+1}, K_{n+1}) = (S_{n+1}, -K_{n+1}) \) we must have \( \kappa \circ \Phi_S = \Phi_S(-S_n, K_n), \)
\[ \kappa \circ \Phi_K = -\Phi_K(-K_n, S_n). \]  
\hspace{1cm} (12)

Thus, \( \Phi_S \) must be an even function of \( K \) while \( \Phi_K \) is an odd function of \( K \). In other words, starting with the most general binomial form for \( \Phi_S(\Phi_K) \), the reflection symmetry condition forces the elimination of all terms containing odd (even) powers of \( K \) in \( \Phi_S(\Phi_K) \). The translation symmetry leads to further restrictions on the form of \( \Phi_S(\Phi_K) \). The interaction functions will be restricted even further when the SO(2) symmetry is considered. The final result, derived by Mureithi et al. (2002) is the following final form for the interaction functions (when \( m \) is odd)
\[ \Phi_S = \begin{bmatrix} p(r_1, r_2, r_3)S + q(r_1, r_2, r_3)\bar{S}^{n-1}K^m \\ r(r_1, r_2, r_3)K + s(r_1, r_2, r_3)\bar{S}^nK^{m-1} \end{bmatrix}. \]  
\hspace{1cm} (13)

In (13), the ‘coefficients’ \( p, q, r, s \) are polynomial functions of the \( \Gamma \)-invariant functions \( r_1 = S\bar{S}, \)
\[ r_2 = K\bar{K}, \quad r_3 = S^n\bar{K}^m, \quad r_4 = \bar{S}^nK^m \]  
and the wavelength ratio \( \lambda_k/\lambda_s = m/n \) with \( m \) and \( n \) relatively prime. An identical approach is used to derive the amplitudes for interactions between the reflection-asymmetric modes \( K, K^1 \). For more details on the derivation of coupled equations based on symmetry, the reader is referred to Golubitsky et al. (1988). Amplitude quations for 3D vortex mode interactions have been derived by Barkley et al. (2000).

2.3 Symmetrical Forcing of the Karman Mode (S,K Interaction)

Physically the case where the cylinder is forced to oscillate in the flow direction (S mode) is considered. Wavelength ratios are \( \lambda_k/\lambda_s = m/n \geq 1 \). For \( m/n = 1 \), the K mode mapping (\( \Phi_K \) in equation (13)) is
\[ K_{n+1} = \left(1 + \alpha_0 + \gamma_1|S_n|^2 + \alpha_2|K_n|^2 \right)K_n + \delta_0|S_n|^2\bar{K}_n \]  
\hspace{1cm} (14)

where the invariant polynomials \( r_s, s, s \) (in equation (13)) have been expanded to give only up to 3rd order terms in the mapping. Equation (14) is the equation governing the dynamics of the Karman mode, \( K \), taking into consideration the effect of the symmetrical mode \( S \). (the corresponding mapping, \( \Phi_S \) governing the dynamics of mode \( S \) can also be similarly written). In the case of forced cylinder excitation, equation (14) is used to describe the response of the Karman mode to inline excitation.

The stability of the non-zero fixed point (\( \bar{K} \)) of (14) is given by the derivative \( f(\bar{K}) = 1 - 2(\alpha_0 | + (\gamma_1 \pm \delta_0)|S|^2) \). For \( S = 0 \), no excitation, \( f(\bar{K}) = 1 - 2|\alpha_0 | < 1 \). This merely confirms the expected result that natural vortex shedding is a stable phenomenon. Limit cycle instability is signaled by \( f(\bar{K}) \) exiting the unit circle. There are
three possible scenarios. Exit along the real axis at $f(K) = 1$ corresponds to a pitch fork instability of the limit cycle. Exit along the real axis at $f(K) = -1$ is associated with a period-doubling instability of the limit cycle. Exit at positions away from the real axis corresponds to a Hopf bifurcation of the limit cycle. In the latter case quasi-periodicity or frequency locking may result. Mureithi et al. (2002) have shown, based on experimental data, that $f(K)$ approaches $-1$ as $S$ increases. This leads to the conclusion that a period doubling bifurcation of the Karman mode occurs. In other words, inline excitation destroys the limit cycle $K$ by inducing a period-doubling bifurcation. Experimental evidence is presented below.

2.4 Asymmetrical Forcing of the Karman Mode ($K_1,K$ Interaction)

Let $K,F$ be the $K,K_1$ modal amplitudes, respectively. Interaction with $\lambda_K/\lambda_{K_1} = m/n = 1$ leads to the following mapping for the $K$ mode

$$K_{n+1} = \left(1 + \alpha_0 + \alpha_1 F_n \right) K_n + \beta_{01} K_{n+1} F_n$$ (15)

Equation (15) is the mode-interaction equation governing the dynamics of the Karman mode amplitude when subject to an excitation mode $K_1$ (having amplitude $F$) of identical symmetry (here, mode $K_1$ is generated by cross-flow excitation of the cylinder). Equation (15) suggests strong low order term coupling between modes $K$ and $K_1$ (geometrically, identical modes in this case). Asymmetric excitation at the Karman frequency (equivalently, wave-number) can be expected to strongly affect vortex shedding.

When the wavelengths are different, thus $\lambda_K/\lambda_{K_1} = m/n$, we have the map

$$K_{n+1} = \left(1 + \alpha_0 + \alpha_2 F_n \right) K_n + \beta_{01} K_{n+1} F_n$$ (16)

the $K_1$ mode (amplitude $F$) now has little effect on the $K$ mode (at low order nonlinear terms). Note that the term $\alpha_1 |F|^2$ is a simple ‘damping’ term, which cannot lead to complex dynamics. Strong lock-in can therefore not be expected. Experimental evidence of these observations is presented below. We finally note that for $\lambda_K/\lambda_{K_1} = m/n = 3$, we have the map

$$K_{n+1} = \left(1 + \alpha_0 + \alpha_3 |F_n|^2 \right) K_n + \beta_{03} F_n^3$$ (17)

3. EXPERIMENTS

3.1 Experimental Setup

Experiments were conducted in a small wind tunnel. A schematic view of the test section is shown in Fig.2. The test section measures $0.10 \times 0.10 \times 0.90$ m. The main set of tests were conducted with a cylinder having diameter $D = 20$ mm. This causes significant blockage in the wind tunnel. A few tests were also done with a $D = 10$ mm cylinder; corresponding to a 50% change in blockage. These tests gave results similar to those obtained with the larger cylinder; thus, the period-doubling instability was also obtained with the smaller cylinder. Hence, although blockage effects certainly alter the local characteristics of the flow,
the global dynamics are not significantly changed (note also that since the symmetry of the system is unchanged, basic (symmetry dependent) results should not change significantly).

Flow in the wind tunnel is generated through suction by an axial fan. A hotwire sensor located 2.25 diameters downstream of the test cylinder (and 0.5D off centerline) picks up local flow oscillations. The Karman mode $K$ is generated by natural vortex shedding. The second mode, ($S$ or $K_1$) is mechanically generated via forced excitation of the test cylinder either parallel to the flow direction ($S$ mode) or transverse to the flow ($K_1$ mode). Tests presented here were conducted at a wind tunnel velocity $V = 1.45\text{ m/s}$; the corresponding Reynolds number varied in the range 1852-1914 depending on ambient temperature. The vortex shedding frequency from the stationary cylinder is $\Omega = 14.5\text{ Hz}$.

In the results presented below, the cylinder is excited at a frequency $\omega$ with amplitude $X_0$. Results are give in terms of the frequency ratio $\omega/\Omega$. Note that $\omega/\Omega > 1$ corresponds to superharmonic excitation while $\omega/\Omega < 1$ corresponds to subharmonic excitation. For all tests, the reference case is that of a stationary cylinder. Results presented include response spectra, coherence and phase variations at various frequencies of interest. We reiterate here, that the goal of these tests was to determine the response characteristics of the Karman mode to external excitation.

3.2 Harmonic Asymmetrical Excitation (Modes $K_1,K$, $\omega/\Omega = 1$)

We consider first the harmonic excitation of the $K$ mode by the $K_1$ mode; thus the test cylinder is oscillated normal to the flow direction at the shedding frequency; $\omega/\Omega = 1$ ($\omega = \Omega = 14.5\text{ Hz}$). The cylinder excitation amplitude is varied in the range $0 - 1.0\text{ mm}$ (corresponding to $0 - 0.05D$).

Typical power spectra are shown in Fig.3 for three excitation amplitudes. Compared to the stationary cylinder case, the peak amplitude is drastically higher. There is a clear transfer of energy from the cylinder to the flow at the Karman frequency. In other words, lock-in has occurred. Additionally when the forcing amplitude $X_0 = 1.0\text{ mm}$ small peaks at $2\omega$ (29 Hz) and $3\omega$ (43.5 Hz) are also observed (Fig.3). This attests to the nonlinear nature of the Karman mode oscillator. The robustness of the corresponding flow structures is confirmed by the high coherence (between cylinder motion and flow perturbations) in Fig.4. The complete test results are summarized in Fig.5. The change in response amplitude (relative to the stationary cylinder case) at the Karman frequency, $\Omega$, as well as at the harmonic $2\Omega$ are plotted in Fig.5(a). The corresponding phase variation and coherence are shown in Figs.5(b,c). The test parameter varied is the cylinder excitation amplitude $X_0$. The peak resonance occurs near $X_0 = 0.6\text{ mm}$ ($=0.12D$). Beyond this excitation level, the results suggest that tube motion has a damping effect on the Karman mode. The phase angle shows a gradual decrease, from 180 degrees, with excitation

![Fig.3](image-url) **Fig.3** Spectra for (a) $X_0=0.4$ mm, (b) $X_0=0.6$ mm and (c) $X_0=1.0$ mm; the frequency ratio $\omega/\Omega=1$.

![Fig.4](image-url) **Fig.4** Coherences for (a) $X_0=0.4$ mm, (b) $X_0=0.6$ mm and (c) $X_0=1.0$ mm; the frequency ratio $\omega/\Omega=1$. 
implications of symmetry in inline and karman shedding mode interaction

amplitude and has a value of 100 degrees at peak resonance. The coherence for the Ω component is close to 1 throughout. Coherence for the Ω component is more variable but high at 0.4 and > 0.8 mm excitation.

3.3 Superharmonic Asymmetrical Excitation (Modes K1, K, \(\omega/\Omega = 3/1\))

Consider next the case \(\omega/\Omega = 3/1\). Results are summarized in Fig.6. In the figure we note that there is low but gradual increase in the Karman mode amplitude. The frequency variation is more interesting, Fig.6(d). Here, the Karman mode frequency is plotted versus cylinder amplitude. Clearly, the frequency gradually decreases with forcing amplitude; from 13.5 Hz for the stationary cylinder to 12.2 Hz for \(X_0 = 1.0\) mm.

3.4 Subharmonic Asymmetrical Excitation (Modes K1, K, \(\omega/\Omega = 1/2, \omega/\Omega = 1/3\))

Next we look at the case where the Karman mode is excited subharmonically. Typical spectra for excitation at \(\omega/\Omega = 1/2 (\omega = 7.25\) Hz, \(\Omega = 14.5\) Hz) are shown in Fig.7. There is signifi-

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Fig.5 Variation of (a) power spectra peaks amps., (b) coherence and (c) phase with forcing amplitude for \(\omega/\Omega=1\). ( - denotes \(\omega: 43.5\) Hz and --- denotes \(2\omega: 29\) Hz.)

Fig.6 Variation of (a) power spectra peaks, (b) coherence, (c) phase and (d) modified Karman frequency, with forcing amplitude for \(\omega/\Omega=3/1\). ( - denotes \(\omega: 43.5\) Hz and --- denotes vortex shedding frequency.)
cant response at the three harmonics $\omega = 7.25$ Hz, $2\omega = \Omega = 14.5$ Hz and $3\omega = 21.75$ Hz. Focusing on the frequency $2\omega = \Omega$ the peak amplitude is found to change significantly. This clearly suggests considerable interaction between the modes. The variation in peak amplitude is plotted in Fig. 8. The peak amplitude relative to the case of a stationary cylinder increases with forcing amplitude. This is also the case for the coherences. The phase is nearly constant for cylinder amplitudes higher than $X_0 = 0.6$ mm. Very similar results are obtained for forcing at $\omega/\Omega = 1/3$. The interaction appears stronger since up to 4th order harmonics are observed. Results for this case are presented in Fig. 9. The excitation amplitude of 0.6 mm marks a threshold point. Harmonics of order 2, 3, and 4 are present. This, together with results in the previous case suggests that the Karman mode is much more ‘compliant’ with subharmonic asymmetric (K1 mode) excitation than superharmonic excitation.

Fig.7 Power Spectra for $\omega/\Omega=1/2$ and (a) $X_0=0.4$ mm, (b) $X_0=0.6$ mm and (c) $X_0=1.0$ mm.

3.5 Harmonic and Super / Subharmonic Symmetrical Excitation

Experiments were also performed with reflection-symmetric S-mode excitation, in which the test cylinder was mechanically oscillated parallel to the flow direction. For excitation with frequency ratio $\omega/\Omega = 1/2$, the Karman mode is significantly attenuated by the S mode. A more detailed analysis of the effect of the symmetrical excitation mode S on the Karman mode K has been presented in Mureithi et al. (2002). We reproduce here some of

Fig.8 Variation of (a) power spectra peaks amplitude, (b) coherence and (c) phase with forcing amplitude for $\omega/\Omega=1/2$. ( --- denotes $\omega: 7.25$Hz and --- denotes $2\omega: 14.5$Hz and denotes $3\omega: 21.75$Hz.)

Fig.9 Variation of (a) power spectra peaks amplitude, (b) coherence and (c) phase with forcing amplitude for $\omega/\Omega=1/3$. ( --- denotes $\omega: 4.875$Hz , --- denotes $2\omega: 9.75$Hz, --- denotes $3\omega: 14.625$Hz, -- denotes $4\omega: 19.5$Hz.)
the pertinent results. For $\omega/\Omega = 1$ typical response spectra are shown in Fig.10. The most important result for symmetrical $S$ mode excitation was the occurrence of a subharmonic at exactly half the forcing ($S$ mode) frequency. Indeed this period-doubled component becomes predominant at high enough excitation amplitudes, Fig.10. This phenomenon persisted also for superharmonic excitation for up to 3 times the Karman frequency excitation. In Fig.11 local velocity rms amplitudes at the forcing frequency $\omega$ and modified Karman frequency $\Omega'$ are shown. Note the drastic drop in perturbation amplitude at the original Karman frequency ($\Omega = \omega$), in spite of the fact that external forcing is input at this frequency. Fig.12 shows the variation of the $K$-mode/$S$-mode frequency ($\Omega'/\omega$) ratio versus the $S$-mode/$K$-mode (fixed cylinder) ($\omega/\Omega$). The square and diamond symbols correspond to different test series. For subharmonic excitation, $\omega/\Omega < 1$, the dominant response appears at the excitation frequency (far removed from the Karman frequency). On the other hand, for superharmonic excitation, the dominant response is at half the forcing frequency. Resonances similar to those observed for $K1$ mode subharmonic (1/2, 1/3) excitation were either absent or very weak when observed.

4. CORRELATION BETWEEN THEORY AND EXPERIMENTS

In this section, we attempt to explain some of the experimental observations, using the theoretical analysis of section 2 as a guide. The main experimental findings are:

(i) Subharmonic ($\omega/\Omega < 1$) asymmetrical (cross-flow) $K1$ mode excitation of the Karman mode, $K$, triggers lock-in at several harmonics of the $K1$ mode frequency.

(ii) Superharmonic ($\omega/\Omega > 1$) asymmetrical (cross-flow) $K1$ mode excitation of the Karman mode has little effect on the Karman mode amplitude, but induces a detuning
effect progressively decreasing the Karman mode frequency.

(iii) Subharmonic (ω/Ω < 1) symmetrical (inflow) S mode excitation of the Karman mode has a damping effect on the Karman mode.

(iv) Harmonic and superharmonic (ω/Ω ≥ 1) symmetrical (inflow) S mode excitation of the Karman mode triggers a period-doubling instability, ‘destroying’ the original Karman mode.

To understand the findings (i) and (ii), we refer to equations (15-17). These equations model the response of the (Karman) K mode to (cross-flow) K1 mode excitation. Based on the structure of these equations, several general conclusions can be drawn. For each equation, the most important coupling term is tabulated in Table 1 below (re: F-term); note that terms corresponding to other K1/K frequency ratios (besides those considered in equations (15-17)) are also included in the table. The order and structure of the F-term is expected to play a key role in determining the expected dynamics. The order of this term is given in column 3; note for instance the F-term is 5th order for 2/1 excitation. More importantly, the order of just the K1 mode component is also shown in the 4th column. The order of the K1 mode may be interpreted as a measure of the effect of external excitation on the Karman mode; thus O(1) indicates strong (coupling) effect while O(4) indicates weak effect. Based on this we note that superharmonic excitation will have less effect on the Karman mode compared to subharmonic excitation. On the other hand, excitation with ratios 1/1 and 1/3 could strongly affect the Karman mode. This general conclusion agrees qualitatively with main experimental findings listed above. Thus, strong resonance response occurs for 1/3 and 1/1 excitation with moderate response for 1/2 excitation. On the other hand, no lock-in is observed for superharmonic 2/1 and 3/1 excitation.

For harmonic/subharmonic (ω/Ω = 1/1, 1/2, 1/3) excitation, significant lock-ins (resonances) were found to occur. Recall that the K mode has Z2(κ, π) symmetry. Relative to this symmetry, the K1 mode will have the following symmetries, Z2(κ, π), Z2(κ, 2π), Z2(κ, 3π) for 1/1, 1/2 and 1/3 frequency ratios respectively; here π refers to a half period spatial shift in K mode wavelength units. Since a phase shift of 2π corresponds to zero phase shift, the K1 mode corresponding to a 1/2 ratio has pure reflection symmetry, Z2(κ), relative to the K mode. On the other hand for the 1/3 ratio K1 mode we have the reduction Z2(κ, 3π) → Z2(κ, π); giving the same symmetry as the K mode itself. The relationships between the K1 and K mode symmetries are shown in the isotropy lattices of Fig.13 for ω/Ω = 1/2 and 1/3. At the top of the

<table>
<thead>
<tr>
<th>ω/Ω</th>
<th>F-term</th>
<th>F-term Order</th>
<th>K1 mode Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2/1</td>
<td>(F_n^4)</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>3/1</td>
<td>(F_n^3)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1/2</td>
<td>(\bar{F}_n^2)</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>1/3</td>
<td>(\bar{F}_n^2)</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1/1</td>
<td>(F_n)</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig.13 Symmetry isotropy lattice for cross-flow excitation and (a) ω/Ω = 1/2, (b) ω/Ω = 1/3, respectively.
lattice is the original symmetry $\Gamma$. Symmetries down the hierarchy are subgroups of symmetries above them (as indicated by arrows). Clearly the $K_1$ and $K$ mode symmetries must be subgroups of $\Gamma$. At the bottom of the hierarchy is the identity, which corresponds to no symmetry. When interaction between two modes occurs, the result is a steady state ‘mixed’ mode solution having only the partial symmetries (subgroups) common to the interacting modes; symmetries (subgroups) not common to both modes are ‘broken’. Since the symmetry $Z_2 (\kappa, 3\pi)$ is common to both $K_1$ and $K$ modes, strong mode (steady state) interaction can be expected for $\omega/\Omega = 1/3$. For $\omega/\Omega = 1/2$ (as well as 2/1) the $K_1$ and $K$ modes have no common symmetries. Consequently, interaction between the modes cannot result in a steady state solution, or alternatively, (steady state) lock-in cannot occur. This same conclusion was previously reached by Williamson and Roshko (1988) by graphically demonstrating the symmetry incompatibilities. More recently, Krishnamoorthy et al. (2001) reported observing a periodic topology in the wake for 1/2 excitation, but noted that a synchronized wake pattern (presumably steady state mixed mode here) did not form, in agreement with the findings here.

Although symmetry limitations do not allow steady state mode locking for 1/2 cross-flow excitation, unsteady solutions, corresponding to traveling waves are allowed. To see this, we consider the polar form of equation (16) together with the corresponding equation for the $K_1$ mode response. Expressing the complex amplitudes in the polar forms $K_n = k_n e^{i\psi_n}, F_n = f_n e^{i\phi_n}$, the mode interaction mappings become

\begin{align}
  k_{n+1} &= (1 + \alpha_0 + \alpha_2 k_n^2 + \alpha_2 k_n^2)k_n + \beta_0 k_n f_n^2 \cos \Theta_n \\
  F_{n+1} &= (1 + \gamma_0 + \gamma_2 f_n^2 + \gamma_2 f_n^2) f_n + \delta_0 k_n^3 f_n^2 \cos \Theta_n \\
  \Theta_{n+1} &= \Theta_n - 4k_n f_n^3 (\delta_0 k_n + \frac{1}{2} \beta_0 f_n) \sin \Theta_n
\end{align}

where $\Theta_n = 2(\phi_n - \psi_n)$. The presence of symmetry reduces the effective number of equations from 4 to 3; the modal phases are not independent. A traveling wave solution is obtained when $(\delta_0 k_n + 1/2 \beta_0 f_n) = 0$. The two modal components appear with a constant phase relationship; note, however, that the individual phases are time dependent. The solution corresponds to a traveling wave drifting at constant speed. For more details on symmetry and traveling waves, the reader is referred to Armbruster et al. (1988) who consider a system with $O(2)$ symmetry.

Returning to the experimental results reported in the literature, it is clear that an organized structure (topology) will be observed and that depending on the speed of the traveling wave, the structure may appear locked-in. In the experimental results reported here, a harmonic response is observed for 1/2 excitation. The test results are however, not conclusive on whether the observed harmonics are due to a traveling wave (unsteady) or steady mode locked state.

All the foregoing results are based on the assumption of small amplitude excitations; the maximum cylinder amplitude tested being 0.05D. Ongoren and Rockwell (1988) found significant lock-in involving vortex coalescence for $\omega/\Omega = 1/2$ and 1/3 $K_1$ mode excitation where the $K$ mode was ‘recovered’ in the far wake; they considered cylinder amplitudes from 0.13D and higher, which are ‘large’ relative to the assumptions of the theory considered here. It is not clear whether or not the $\omega/\Omega = 1/2$ ‘lock-in’ is related to the traveling wave solution discussed above.

For reflection-symmetric $S$ mode excitation, we refer the reader to our previous work (Mureithi et al. 2002). There, it is proved that period-doubling instability is the predominant dynamical phenomenon to be expected for $S/K$ mode interaction with $1 \leq \omega/\Omega \leq 2$. 
Finally we note that our present Poincare-map based analysis cannot explain the frequency drift induced by superharmonic \( K1 \) mode excitation partly because taking a Poincare map invariable makes the analysis ‘blind’ to dynamics between intervals of the discrete map. An alternative approach is being considered.

5. BIFURCATION CONTROL OF THE KARMAN MODE

The unforced Karman mode (amplitude) satisfies (to 3rd order) the equation,

\[
K_{n+1} = \left(1 + \alpha_0 + \alpha_2 |K_n|^2 \right) K_n \tag{19}
\]

We now consider the possibility of controlling this system by introducing the symmetrical mode with a controlled amplitude \( S \). The control effect of mode \( S / K1 \) on mode \( K \) will be governed by equation (14 / 15). In effect, equation (14 or 15) gives an approximation of the expected spatio-temporal dynamics when control is applied. It is important to note, that the controlled mode \( K \) will in turn affect the controlling mode \( S \) or \( K1 \). Recall that in actual control experiments, control is introduced locally in space. The resulting control perturbations thereafter interact freely with the mode to be controlled.

An approach to the control of this system is via first considering the fixed point solutions of (19). The zero fixed point corresponds to the unper-turbed steady state (prior to the onset of vortex shedding). The stability of this state is determined by the parameter \( \alpha_0 \). Recall that this parameter is a measure of the ‘distance’ from the critical Reynolds number for Karman shedding initiation, i.e. \( \alpha_0 \propto (Re - Re_c) \). The non-zero fixed point corresponds to shedding having limit cycle amplitude \( |K| = \frac{-\alpha_0}{\alpha_2} \). Bifurcation control attempts to ‘delay’ the onset of the instability leading to the Karman mode; in other words, maintain the stability of the zero fixed point over as wide a Reynolds number range as possible by delaying the bifurcation.

5.1 Feed-Forward Bifurcation Control of Karman Mode via S-mode Forcing

Physically we consider the case where symmetrical perturbations are introduced in the flow in the flow direction (S mode). Wavelength ratios are \( \lambda_K / \lambda_S = m \geq 1 \). For \( m = 1 \), the \( K \) mode mapping in polar form, with \( K = re^{i\phi} \), is

\[
\begin{align*}
    r_{n+1} &= \left(1 + \alpha_0 \right) r_n + \alpha_2 r_n^3 + u_r \\
    \phi_{n+1} &= \phi_n + u_\phi
\end{align*}
\tag{20}
\]

The control input is given by

\[
\begin{align*}
    u_r &= \left( \gamma_{11} q^2 - \delta_{01} q^2 e^{2i\phi} \cos 2\phi_n \right) r_n \\
    u_\phi &= -\delta_{01} q^2 e^{2i\phi} r_n \sin 2\phi_n
\end{align*}
\tag{21}
\]

where we have used the definition \( S = q e^{i\phi} \). Equation (21) shows that true independent feedforward control is impossible in this spatio-temporal system; thus, any control input becomes coupled to the \( K \) mode itself. The stability of the zero fixed point of (20) is determined by the quantity

\[
f_r(0) = 1 + \alpha_0 + (\gamma_{11} - \delta_{01} e^{2i\phi}) q^2 \tag{22}
\]

The stability of the zero fixed point (steady unperturbed flow) requires that \( |f_r(0)| < 1 \). Mureithi et al. (2002) have shown experimentally that the quantity \( (\gamma_{11} - \delta_{01}) > 0 \). If \( \gamma_{11} < 0 \), it is clear that the zero fixed point can be stabilized for all values of \( \delta_{01} \). On the other hand, if \( \gamma_{11} > 0 \) stabilization, using the controller (21), is only possible if \( \delta_{01} < -\gamma_{11} \). The sign \( \gamma_{11} \) of needs to be determined experimentally.

5.2 Feed-Forward Bifurcation Control of Karman Mode via K1-mode Excitation

We consider here the control of the 3rd order \( K \)
mode using perturbations of identical symmetry, \( Z_2(\kappa, \pi) \). The unforced \( K \) mode with external excitation becomes
\[
K_{n+1} = \left( 1 + \alpha_0 + \alpha_2 |K_n|^2 \right) K_n + u(K_n, F_n) \tag{23}
\]
where the control \( u(\ldots) \) takes the form
\[
u = \left[ \alpha_1 |F_n|^2 + \alpha_2 K_n F_n \right] K_n + \left[ \beta_0 + \beta_1 |F_n|^2 + \beta_2 |K_n|^2 + \beta_3 K_n F_n \right] F_n \tag{24}
\]
for \( \lambda_K/\lambda_{K_1} = 1 \). In this case, the magnitude and phase of \( F \) can be varied independently to achieve the control goal. Once again the functional form \( u(\ldots) \) is predetermined by the nature of mode interaction between modes having symmetries, \( Z_2(\kappa, \pi) \) and \( Z_2(\kappa, m\pi) \). The simplest controller involving 1st order terms in \( F \) (and \( K \)) takes the form
\[
u = \beta_0 F_n \tag{25}
\]
Bifurcation of equation (23) occurs at \( \alpha_0 = 0 \) without control. To delay the bifurcation, it is clear that proper phasing of \( F \) relative to \( K \) is necessary. Thus some form of feedback (or phase tracking) is therefore needed. Simple feed-forward control is unlikely to work. Note that since the \( K \) and \( K_1 \) modes are coupled at all orders (for \( m = 1 \)), it is practically impossible to get a single correct forcing amplitude to delay the instability. On the contrary, it would seem that the \( K_1 \) mode (without feedback) is more likely to trigger vortex shedding. Indeed this is the result obtained in the experiments (Fig.5).

5.3 Feedback Bifurcation Control of Karman Mode via K1-mode Excitation

The Karman mode, subjected to control now becomes,
\[
K_{n+1} = \left( 1 + \alpha_0 + \alpha_2 |K_n|^2 \right) K_n + u(K_n, F_n(K_n)) \tag{26}
\]

The excitation mode is now determined by the Karman mode. The problem thus reduces to the standard control problem considered by numerous investigators to date. In the first order controller \( u = \beta_0 F_n \) we set the controlled excitation amplitude to
\[
F_n = -\frac{\alpha_0}{\beta_0} K_n. \tag{27}
\]
This controller will work close to the bifurcation point (Reynolds number) at which vortex shedding commences. This can be easily confirmed by looking at the purely linear part of the system (26) with controller (27). For general feedback control (up to higher orders) we may consider the law
\[
F_n = \eta K_n \tag{28}
\]
where the parameter \( \eta \) enables us to set the proper magnitude and phase of \( F \) relative to \( K \). The general controller \( u \) now becomes
\[
u = \left[ \alpha_1 |K_n|^2 + \alpha_2 |K_n|^2 \right] K_n + \left[ \beta_0 \eta + \beta_1 |\eta| |K_n|^2 + \beta_2 \eta |K_n|^2 + \beta_3 \eta^2 |K_n|^2 \right] K_n \tag{29}
\]
The performance of this controller is strongly dependent on the sign and magnitude of the various constants \( \alpha_i, \beta_i \). Consider the case where the ‘control’ mode \( K_1 \) has the same symmetry as the \( K \) mode, \( Z_2(\kappa, m\pi) \) with \( m = 1 \). In this case we expect the following relations to hold between the signs (+/-) of the various coefficients
\[
\text{sgn}(\beta_0) = \text{sgn}(\alpha_0) \quad \text{sgn}(\alpha_1) = \text{sgn}(\alpha_2) = \text{sgn}(\alpha_3) = \text{sgn}(\beta_1) \quad \text{sgn}(\beta_2) = \text{sgn}(\beta_3) \tag{30}
\]
This simply represents the argument that same order terms of \( F \) and \( K \) should have similar effects. Choosing \( \eta < 0 \) is necessary in order to stabilize the zero fixed point at linear order. Note, however, that in this case the term \( \beta_2 \eta |K_n|^2 \) is a destabilizing term. This result may explain why, even with

Implications of Symmetry in Inline and Karman Shedding Mode Interaction
'proper' phasing, complete elimination of the K mode is rarely achieved.

6. LIMIT CYCLE CONTROL OF THE KARMAN MODE

In Section 5 we have considered control aimed at maintaining the stability of the zero fixed point; physically corresponding to delaying the onset of the Karman mode shedding. The foregoing analysis, as well as experimental evidence by other investigators show, that such bifurcation control will only be possible over a limited range of Reynolds numbers beyond the critical Reynolds number $Re_c$.

In this section, we consider the alternative approach for Reynolds numbers significantly different from $Re_c$. The goal now is to attempt to destabilize the existing Karman mode; in our approach, we try to change the stability of the non-zero fixed point; from stable to unstable. This involves inducing a bifurcation of the fixed point. Here we do not consider in detail the resulting dynamics.

6.1 Feed-forward Control of Karman Mode via Symmetrical S-mode Excitation

The dynamical system under consideration is given by

$$
\begin{align*}
\dot{r}_n &= (1 + \alpha_0) r_n + \gamma_2 S_n^2 + u_r
\end{align*}
$$

$$
\phi_{n+1} = \phi_n + u_\phi
$$

with appropriate control (equation (21)). The non-zero fixed point is

$$
r^2 = \frac{-\sigma \pm \mu}{\gamma}
$$

(32)

This then is the limit cycle amplitude. In the equation above, $\sigma = (\alpha_0 + \gamma_1 |S|^2)$, $\mu = \delta_0 S^2$, $\gamma = \gamma_2$. The stability derivative for this Karman mode limit cycle is therefore

$$
f_r(r) = 1 - 2(|\alpha_0| + (\gamma_1 \pm \delta_0) S^2)
$$

(33)

For $S=0$, no control, $f_r(r) = 1 - 2|\alpha_0| < 1$. This merely confirms the expected result that natural vortex shedding is a stable phenomenon. Limit cycle instability is signaled by $f_r$ exiting the unit circle. There are three possible scenarios. Exit along the real axis at $f_r = 1$ corresponds to a saddle node instability of the limit cycle. Exit along the real axis at $f_r = -1$ associated with a period-doubling instability of the limit cycle. Exit at away from the real axis corresponds to a Hopf bifurcation of the limit cycle. In this latter case, Quasi-periodicity of frequency locking may result. In the present work, the goal of limit cycle bifurcation control is to drive the stability derivative $f_r$ outside the unit circle. The result, as discussed above, is the destruction of the limit cycle. Physically, this would correspond to interruption (or destruction) of the period Karman shedding. Mureithi et al. (2002) have shown, based on experimental data, that $f_r$ approaches $-1$ as $S$ increases. This leads to the conclusion that a period doubling bifurcation of the Karman mode occurs. In other words, the control effect now is to destroy the limit cycle K by inducing a bifurcation. It is important to note, however, that the result is not a steady (unperturbed flow) but rather a flow oscillating at half the original shedding frequency. We consider this to be successful control, from a fluid-structure interaction view point; any lock-in resonance that may have occurred at the Karman frequency $\Omega$ will certainly not occur at half the frequency $\Omega/2$.

7. CONCLUSIONS

The dynamics of a forced cylinder wake have been examined both theoretically and experimentally. Symmetrical (S mode) excitation was found to strongly affect the Karman (K) mode by trigger-
ing a subharmonic instability; this effect occurs for frequencies as high as 3 times the Karman frequency. Asymmetric (mode $K_1=K$) forcing at the Karman frequency enhances the Karman mode. Forcing at higher frequencies (with mode $K_1\neq K$) has little effect on the Karman mode, however, a detuning effect was found. Subharmonic asymmetrical excitation triggered response at harmonics of the excitation mode $K_1$. Steady state response was shown to consist of modes having subgroup symmetries of the fundamental Karman mode. In essence, lock-in was shown to be possible whenever the symmetry group of the excitation mode $K_1$ was a subgroup of the Karman mode symmetry group $Z_2(\kappa,\pi)\times S^1$. Based on this, it is suggested that ‘lock-in’ reported for 1/2 frequency ratio excitation may correspond to a traveling wave topology rather than a steady state.

Preliminary work on Karman mode control has also been presented. Two control strategies were considered. (Anti-)bifurcation control to delay the onset of vortex shedding and limit cycle control following the onset of vortex shedding. Anti-bifurcation control was shown to have limited capability. Limit cycle control (destruction) was demonstrated using reflection symmetric forcing.

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